Agribusiness Analysis and Forecasting

Multivariate Normal

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Joint versus Marginal Distributions

Thus far, we have been thinking about each variable in a system individually. We have been thinking about each variable's univariate distribution, a.k.a, its *marginal distribution*. We have been implicitly assuming realizations of random variables occur *independently* of one another.

In this topic, we start thinking about the *joint distribution* (a.k.a., multivariate distribution) of a system of variables. This is a single probability distribution for all of a system's variables that accommodates some form of *dependence* among the variables.

Different Joint Distribution Approaches

* Multivariate Normal distribution (this topic). All variables have a normal marginal distribution.

* Mixed marginals where each variable has a different marginal distribution (next topic). For example:

- $Y_1 \sim$ Uniform
- $Y_2 \sim$ Normal
- $Y_3 \sim$ Empirical
- $Y_4 \sim$ Beta

Why use joint distributions?

Data are generated contemporaneously.

- Price and yield are observed each year for related commodities.
- Corn and sorghum used interchangeably for animal feed so prices are related.
- Steer and heifer prices are related.
- Yields of crops on the same farm have the same weather conditions.

Supply and demand forces affect prices similarly, bear market or bull market; prices move together.

- Prices for tech stocks move together.
- Prices for an industry or sector's stocks move together.

Why use joint distributions?

If correlation is ignored when random variables are correlated, results are biased.

Suppose $\tilde{Y}_3 = \tilde{Y}_1 + \tilde{Y}_2$ and the model is simulated without correlation, assuming $\rho_{1,2}$=0:

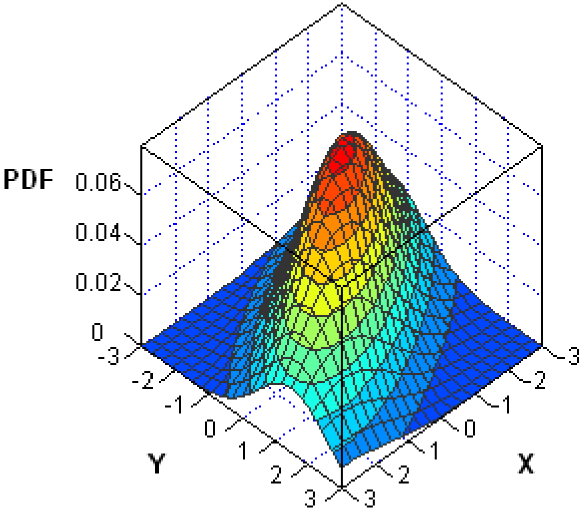
- Mean is unaffected.
- If the true $\rho_{1,2} > 0$ then the model will understate the risk for Y_3.
- If the true $\rho_{1,2} < 0$ then the model will overstate the risk for Y_3.

If $\tilde{Y}_3 = \tilde{Y}_1 * \tilde{Y}_2$:

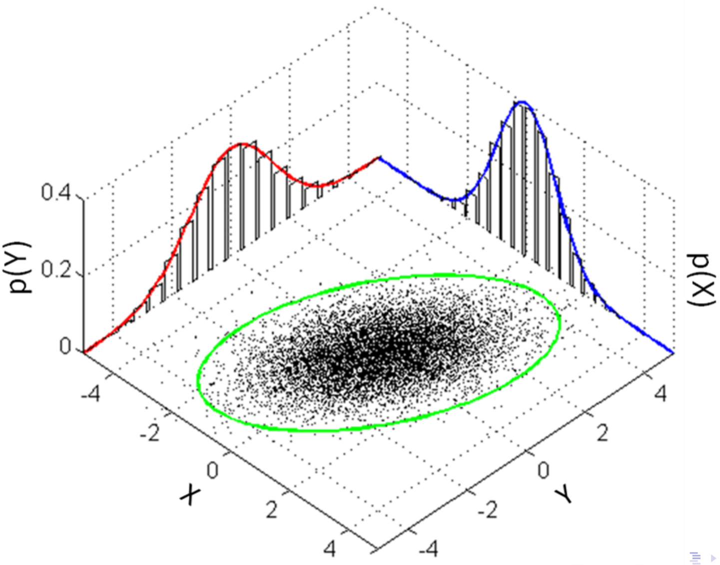
- Not only the variance, but the mean of Y_3 is biased, as well.

Refer to the technical appendix for detailed analysis.

MV Normal Joint PDF



MV Normal Marginal Distributions



Reminder...

You should only be simulating i.i.d random variables.

- $E(Y_t) = E(Y_{t-1}) = \mu$ (no trends, no cycles)
- $\sigma_t^2 = \sigma_{t-1}^2 = \sigma^2 < \infty$
- No autocorrelation

The parameters above are not a function of time.

Generating Draws for MV Normal

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} + \sqrt{\begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ & \sigma_2^2 & \sigma_{23} \\ & & \sigma_3^2 \end{bmatrix}} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \tag{1}$$

- The z_i are independent standard normal draws.
- μ_i is the mean for the i^{th} variable in the system.
- σ_{ij} is the covariance between the i^{th} variable and the j^{th} variable.
- $\sqrt{\ }$ denotes the *Cholesky decomposition* of the covariance matrix.

Simulating MVN in Simetar

For 4 random variables...

- This is an array formula
- Start by highlighting 4 cells where the result will go
- Then
  =MVNORM(4x1 Means Vector, 4x4 Covariance Matrix)
  =MVNORM(A1:A4 , B1:E4)
  Control Shift Enter

Limitations to MV Normal

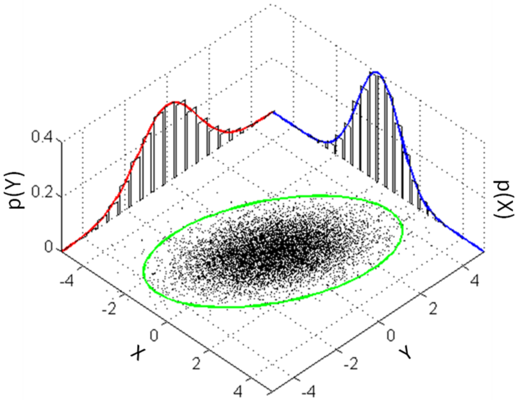
- Under MV Normal, all marginal distributions in the system are normal.

  But we won't generally have control over which marginal distributions are appropriate for the variables in our system.

- Under MV Normal, dependence among variables is strictly linear.

  But dependence among variables is often non-linear.

MV Normal Marginal Distributions



$$\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 1 & 0.6 \\ & 2 \end{bmatrix}$$

Technical Appendix Example: $\tilde{Y}_3 = \tilde{Y}_1 + \tilde{Y}_2$

Recall:

$$\rho_{(Y_1, Y_2)} = \frac{\text{Cov}(Y_1, Y_2)}{\sigma_{Y_1} \cdot \sigma_{Y_2}},$$
$$\sigma_Y = \sqrt{\text{Var}(Y)}.$$

Expected Value and Variance for the relationship:

- **Expected value:**

$$E(\tilde{Y}_3) = E(\tilde{Y}_1) + E(\tilde{Y}_2) \tag{2}$$

- **Variance:**

$$\text{Var}(\tilde{Y}_3) = \text{Var}(\tilde{Y}_1) + \text{Var}(\tilde{Y}_2) + 2\text{Cov}(\tilde{Y}_1, \tilde{Y}_2) \tag{3}$$

Technical Appendix Example: $\tilde{Y}_3 = \tilde{Y}_1 + \tilde{Y}_2$

Suppose $\rho_{1,2} = -0.4$, the variance of Y_1 is 1.44, and the variance of Y_2 is 100. If you ignore the correlation and assume it is 0:

- The mean is unbiased.
- The variance (risk) is biased:
  - Calculated risk $= 1.44 + 100 = 101.44$.
  - True risk $= 1.44 + 100 + 2 \times (-0.4) \times 1.2 \times 10 = 91.84$.

Thus, ignoring the correlation:

- If $\rho_{1,2} < 0$, you will overstate the risk.
- If $\rho_{1,2} > 0$, you will understate the risk.

Technical Appendix Example: $\tilde{Y}_3 = \tilde{Y}_1 \times \tilde{Y}_2$

Expected Value and Variance for the relationship:

- **Expected value:**

$$E(\tilde{Y}_1 \times \tilde{Y}_2) = E(\tilde{Y}_1) \times E(\tilde{Y}_2) + \text{Cov}(\tilde{Y}_1, \tilde{Y}_2) \tag{4}$$

- **Variance:**

$$\text{Var}(\tilde{Y}_1 \times \tilde{Y}_2) = E(\tilde{Y}_1^2) \times \text{Var}(\tilde{Y}_2) + E(\tilde{Y}_2^2) \times \text{Var}(\tilde{Y}_1) + 2 \times \text{Cov}(\tilde{Y}_1, \tilde{Y}_2) \times E(\tilde{Y}_1) \times E(\tilde{Y}_2) \tag{5}$$

Thus, ignoring the correlation:

- If $\rho_{1,2} < 0$, both mean and variance are biased. You will overstate both mean and the risk.
- If $\rho_{1,2} > 0$, both mean and variance are biased. You will understate both mean and the risk.